SPRING 2024: BONUS PROBLEM 3 SOLUTION

For the matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, write its characteristic polynomial as $p_A(x) = x^3 + c_1x^2 + c_2x + c_3$.

1. Express c_1 and c_3 in terms of the trace and determinant of A. (2 points)

2. Express c_2 in terms of the matrix A' from the lecture of February 28. A' is called the *adjugate* or *classical adjoint* of A. (2 points)

Solution. We first note that, an easy calculation gives

$$\begin{aligned} p_A(x) &= x^3 - (a+e+i)x^2 + \{(ei-hf)+(ai-gc)+(ae-bd)\}x - (afh - aei - bfg + bdi - cdh + ceg) \\ &= x^3 - \mathrm{tr}(A)x^2 + (|A_{11}| + |A_{22}| + |A_{33}|)x - \det(A) \\ &= x^3 - \mathrm{tr}(A)x^2 + \mathrm{tr}(A')x - \det(A), \end{aligned}$$

where A' is the classical adjoint of A. Thus, $c_1 = -\mathrm{tr}(A)$, $c_2 = \mathrm{tr}(A')$, and $c_3 = -\det(A)$.